RESEARCH STATEMENT

EDOUARD FEINGESICHT

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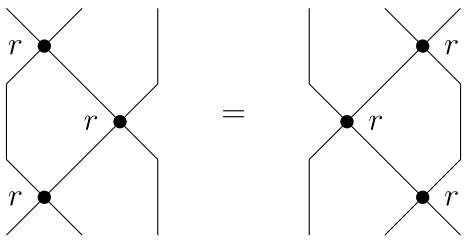
My research is focused on Garside theory, in particular using the Garsideness of structure groups associated to set-theoretical solutions of the Yang–Baxter equation to progress on their classification. This classification also involves many different algebraic structures : braces, representations, algebras over a ring, etc. In general, I try to adapt techniques and objects defined for Coxeter group to Yang–Baxter structure group.

In what follows, the first section lays the general context, the second one summarizes my first article ([Fei24]), the third and fourth focus on some of my current work, and the last the future work and how I'd would approach them.

1. Context: the Yang–Baxter Equation

The Yang–Baxter Equation is a fundamental equation in Statistical Physics, and finding its solutions is still a very active research area. In 1992, Drinfeld ([Dri92]) proposed to first study a particular case: set-theoretical solutions, that is a pair (X, r) with X a set and $r\colon X \times X \to X \times X$ a bijection respecting the braid equation $(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$:



Since then, many advances have been made, and Etingof-Schedler-Soloviev [ESS99] was an important step: the authors proposed to consider set-theoretical solutions that are involutive ($r^2 = \mathrm{id}$) and non-degenerate (writing $r(x, y) = (\lambda_x(y), \rho_y(x))$, for all x in X, λ_x and ρ_x have to be bijective). Moreover, they defined a fundamental notion for the study of solutions: the structure group associated to a solution, defined by the presentation $\langle X \mid xy = \lambda_x(y)\rho_y(x)\rangle$. From the structure group, many other objects and properties have been studied: algebras (Gateva-Ivanova in [Gat04]), I-structures (Gateva-Ivanova or Dehornoy in [GV98; Deh15]), etc. In particular, very important objects are braces, introduced by Rump in [Rum07], which are a sort of "additive" ring-like extra-structure added on groups (here our case structure group or quotients of it). They are currently one of the most-studied approaches to classifying solutions (see for instance the work of Bachiller in [Bac18]), especially finite braces which, in a sense, come from quotients of solutions. Most importantly for us, Chouraqui showed in [Cho10] that this group has the very particular property of being a Garside group, just like Artin groups of spherical type (when the associated Coxeter group is finite). This result was also re-obtained by Dehornoy in [Deh15], and this article was the basis for my PhD thesis. In particular, one of my goals is to have a better understanding of the analoguous properties of the structure groups and spherical Artin groups from a Garside perspective, in hope to help for the classification of finite involutive non-degenerate set-theoretical solutions of the Yang–Baxter equation (which from here on will just be called solutions).

2. Dehornoy's class and Germs

In [Deh15] a positive integer d was associated to any solution, which we call the Dehornoy class, and a "nice" quotient was defined from it (quotienting by "twisted powers" $x^{[d]}$). This quotient is a Garside germ, which is analogous to Coxeter Groups obtained by quotient spherical Artin groups by $s^2 = 1$ for all generators s. The proof of its existence relied on a theorem of Rump stating that finite left non-degenerate solutions are right non-degenerate ([Rum05]), and its bounds and values was not studied. In [Fei24], one of the goals is to obtain the existence without Rump's theorem (and also reproving Rump's theorem after) and study the class from a combinatorial and numerical perspective. Its existence is obtained with a better bound in general ($(n!)^n$ compared to $(n^2)!$) and a conjecture is stated:

Conjecture ([Fei24])**.** *Let S be a cycle set of size n. The Dehornoy's class d of S is bounded above by the "maximum of different products of partitions of n into distinct parts" and the bound is minimal, i.e.*

$$d \leq \max\left(\left\{\prod_{i=1}^{k} n_i \,\middle|\, k \in \mathbb{N}, 1 \leq n_1 < \cdots < n_k, n_1 + \cdots + n_k = n\right\}\right).$$

This conjecture was obtained through the use of representation theory and algorithmics applied to the existing enumeration of solutions upto size 10 by Akgün-Mereb-Vendramin ([AMV22]). Recall that a solution is called square-free if $r(x,x) = (x,x)$ for all x in X, and the permutation group is defined as the subgroup of $\mathfrak{S}_n$ generated by the λ_x. Under some conditions the conjecture was also shown to hold:

Proposition ([Fei24])**.** *If S is square-free and its permutation group $\mathcal{G}$ is abelian then the conjecture holds.*

In the last part of the paper, the focus is on the germ and its Sylow-subgroups. The main result states how, considering the Zappa–Szèp product of germs, we can "decompose" solutions into smaller ones :

Theorem ([Fei24])**.** *Any finite involutive non-degenerate solution can be constructed from the Zappa–Szép product of the germs of cycle sets of class a prime power.*

These two statements aim to reduce the goal of classifying all finite braces to a simpler one:

First, the existence of the germs ([Deh15]) (who determine the solution), which are very particular braces, indicates that we can restrict to braces with "additive" group isomorphic to some $(\mathbb{Z}/d\mathbb{Z})^n$. The integer n determines the size of the solution and d its class, so it is fundamental to understand the behaviour of the class. In particular, having a sharp bound on d gives a large restriction to which braces should be considered for the classification, and this is precisely the point of the conjecture.

Then, the theorem indicates that one notion of "basic" solutions are the ones where the class d is a prime power. Another well-studied notion of "basic" solutions are indecomposable ones (where no proper subset of X is stable by r). Combining these two notions, it is shown in [Fei24] that even more "fundamental" solutions should be the one where the size and the class are powers of the same prime. Moreover, it is also conjectured in [Fei24] that in the case of indecomposable solutions, the class d is bounded by n. Putting all of this together, and assuming the two conjectures, one could hope to restrict to classifying braces with additive group $(\mathbb{Z}/p^a\mathbb{Z})^{p^b}$ with $a \leq b$.

3. Hecke algebras

Continuing on drawing parallels between structure groups and spherical Artin groups, one of my current work consists in defining and studying a Hecke algebra of solutions. In the case of spherical Artin groups, this is a sort of intermediate algebra between the ring algebras of the Artin and the Coxeter group, from which many combinatorial and representation theoretical properties can be obtained (see for instance the book of Geck-Pfeiffer [GP00]). A very general definition is currently obtained and derived from a graphical interpretation of the structure group.

Let (X, r) be a solution of size n ($X = \{x_1, \dots, x_n\}$), class d, structure group G and "augmented" germs $\overline{G}_l$ ($= G/(x_i^{[ld]})$ which is isomorphic to l^n copies of the germ $\overline{G} \times (\mathbb{Z}/l\mathbb{Z})^n$), then we have the following:

Theorem (To appear)**.** *Let $R = \mathbb{Z}[q_1^{\pm1}, \dots, q_l^{\pm1}]$ and $P(X) = (X - q_1)\dots(X - q_l) \in R[X]$. Then $\mathcal{H}(X, P) \coloneqq R[G]/(P(x_i^{[d]})_{1\leq i\leq n}$ has the same dimension as $R[\overline{G}_l] = R[G/(x_i^{[ld]})_{1\leq i\leq n}]$ (precisely $(dl)^n$).*

Moreover this algebra is absolutely semi-simple and $\iota\colon R \to \mathcal{R}$ sending q_i to q_i^{-1} extends to an anti-involution of $\mathcal{H}(X, P)$ by sending x_i to x_i^{-1}.

In particular, we can obtain the exact equivalent of the generic Iwahori-Hecke algebra of a Coxeter group by taking $\mathcal{H} = \mathbb{Z}[q][G]/(x^{[2d]}) = (q - 1)x^{[d]} + q$.

This is just the beginning of an article (hopefully to be finished before the end of my PhD), and I hope that this representation theory/algebras approach could lead to interesting results helping for the classification of solutions.

4. Indecomposability and Irreducibility

A solution (X, r) to the Yang–Baxter equation is called decomposable if there exists a proper decomposition $X = Y \sqcup Z$ such that both $Y \times Y$ and $Z \times Z$ are stable under r, otherwise it is called indecomposable. In [Deh15] a monomial representation θ of the structure group is defined, and was studied in [Fei24]. The goal in this work is to relate the indecomposability of a solution with properties of the monomial representation. In this sense, the following was obtained with Carsten Dietzel and Sylvia Properzi from VUB (Brussels) :

Theorem (To appear)**.** *Let (X, r) be a solution to the Yang–Baxter equation of size at least 3. Then it is indecomposable if and only if θ is an irreducible representation.*

One interest of this statement is that it gives a new direction to study indecomposable solution: it is known that an irreducible monomial representation is induced by a character of a subgroup. Thus, understanding some subgroup and character that induce the representation may lead to new ways to generate indecomposable solutions. This is some current work in collaboration with C. Dietzel and S. Properzi.

5. Future work

In general, my interests lie in anything related to algebra: group theory, representation theory, combinatorics, homology theory, category theory, etc. and I try to make use of the different suitable techniques from these areas when needed. Thus, I would be greatly interested in working on new questions and collaborating with new people, as a way to continually learn and study mathematics while bringing new technics from one domain to another. In particular, here is a list of questions which I am currently working on:

5.1. Bounding the class

The goal would be to prove the conjecture on the sharp bounds of the class. For this, I have two different approaches:

a) Focusing on a brace approach of the permutation group, especially it's additive structure. This would involve giving an explicit characterization of its exponent, or the order of elements.
b) Showing the conjecture for indecomposable solutions ($d \leq n$) and understanding how decomposing a solution affects the class. This approach, may also lead to a better comprehension on how to construct decomposable solutions.

This work, and in general the study of solutions, is a great opportunity to work with other people. In particular, I've a lot of contact with Leandro Vendramin's team at VUB (Brussels, Belgium), and hope that this would be fruitful in producing new work.

5.2. Studying the Hecke algebra

It seems natural, after defining the algebra, to define and study the Schur multiplier and the Kazhdan–Lusztig polynomials associated. This would be in the continuity of adapting the techniques from [GP00]. First restricting to particular families of solutions would be a first step. The difficult part in the general case would be that adapting the technics from [GP00] may be highly computational and more complicated (as we have more data than in the Coxeter case), but the proofs found so far are quite simpler (for instance of the Theorem in section 3), so the introduction of new tools and technics (in particular coming from the study of solutions) should make this project feasable.

This would hopefully lead to collaboration with people working on Coxeter groups and/or Hecke algebras, as there is still a lot to be done.

5.3. Generalization to Weyl groups

The idea would be to adapt the techniques from [Deh15; Fei24] to Weyl group (as the Yang–Baxter case is for the symmetric group). This would involve, for me, to study more Algebraic Group and Geometric Group theory, in particular Weyl groups acting on maximal torii. In particular, this could also lead to a generalization of braces (replacing the additive structure with a Coxeter one, a particular case of regular subgroup of the holomorphy group), and would be very interesting from a Garside perspective, as it would provide a Garside framework to encompass both Weyl groups and Yang–Baxter structure groups.

This question would be a nice opportunity to work with people focused on Geometric Group Theory aswell as those working on braces. Moreover, this would directly lead to generalizations of all the questions mentionned above, thus a very large project encompassing many different objects and technics.

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