RESEARCH STATEMENT

EDOUARD FEINGESICHT

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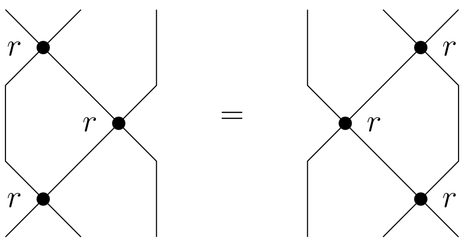
My research is focused on Garside theory, in particular using the Garsideness of structure groups associated to set-theoretical solutions of the Yang–Baxter equation to progress on their classification. This classification also involves many different algebraic structures : braces, representations, algebras over a ring, etc. In general, I try to adapt techniques and objects defined for Coxeter group to Yang–Baxter structure group.

In what follows, the first section lays the general context, the second one summarizes my first published article ([Fei24a]), the third and fourth focus two recent prepublications, and the last is on the future work and how I'd would approach some problems.

1. Context: the Yang–Baxter Equation

The Yang–Baxter Equation is a fundamental equation in Statistical Physics, and finding its solutions is still a very active research area. In 1992, Drinfeld ([Dri92]) proposed to first study a particular case: set-theoretical solutions, that is a pair (X, r) with X a set and $r\colon X \times X \to X \times X$ a bijection respecting the braid equation $(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$:



Since then, many advances have been made, and a seminal paper of Etingof–Schedler–Soloviev [ESS99] was an important step: the authors proposed to consider set-theoretical solutions that are involutive ($r^2 = \mathrm{id}$) and non-degenerate (writing $r(x, y) = (\lambda_x(y), \rho_y(x))$, λ_x and ρ_x have to be bijective for all x in X). Moreover, they defined a fundamental notion for the study of solutions: the structure group associated to a solution, defined by the presentation $\langle X \mid xy = \lambda_x(y)\rho_y(x)\rangle$. From the structure group, many other objects and properties have been studied: algebras (Gateva–Ivanova in [Gat04]), I-structures (Gateva–Ivanova or Dehornoy in [GV98; Deh15]), etc. In particular, very important objects are braces, introduced by Rump in [Rum07], which are a sort of "additive" ring-like extra-structure added on groups (in our case the structure group or quotients of it). They are currently one of the most-studied approaches to classifying solutions (see for instance the work of Bachiller in [Bac18]), especially finite braces which, in a sense, come from quotients of solutions. Most importantly for us, Chouraqui showed in [Cho10] that this group has the very particular property of being a Garside group, just like Artin groups of spherical type (when the associated Coxeter group is finite). This result was also re-obtained by Dehornoy in [Deh15], and this article was the basis for my PhD thesis. In particular, one of my goals is to have a better understanding of the analogous properties of the structure groups and spherical Artin groups from a Garside perspective, in hope to help for the classification of finite involutive non-degenerate set-theoretical solutions of the Yang–Baxter equation (which from here on will just be called solutions).

2. Dehornoy's class and Germs

In [Deh15] a positive integer d was associated to any solution, which we call the Dehornoy class, and a "nice" quotient was defined from it (quotienting by "twisted powers" $x^{[d]}$). This quotient is a Garside germ, which is analogous to Coxeter Groups obtained by quotienting spherical Artin groups by s^2 for all generators s. The proof of its existence relied on a theorem of Rump stating that finite left non-degenerate solutions are right non-degenerate ([Rum05]), and its bounds and values was not studied. In [Fei24a], one of the goals is to obtain the existence without Rump's theorem (and also reproving Rump's theorem after) and study the class from a combinatorial and numerical perspective. Its existence is obtained with a better bound in general ($n!$ compared to $n^2!$) and a conjecture is stated:

Conjecture ([Fei24a]). *Let (X, r) be a solution of size n. The Dehornoy's class d of (X, r) is bounded above by the "maximum of different products of partitions of n into distinct parts" and the bound is minimal, i.e.*

$$d \leq \max \left(\left\{ \prod_{i=1}^{k} n_i \middle| k \in \mathbb{N}, 1 \leq n_1 < \cdots < n_k, n_1 + \cdots + n_k = n \right\} \right).$$

This conjecture was obtained through the use of representation theory and algorithmics applied to the existing enumeration of solutions up to size 10 by Akgün–Mereb–Vendramin ([AMV22]).

Recall that a solution is called square-free if $r(x, x) = (x, x)$ for all x in X, and the permutation group $\mathcal{G}$ is defined as the subgroup of $\mathfrak{S}_n$ generated by the λ_x. Under some conditions the conjecture was also shown to hold:

Proposition ([Fei24a]). *If (X, r) is square-free and its permutation group $\mathcal{G}$ is abelian then the conjecture holds.*

In the last part of the paper, the focus is on the germ and its Sylow subgroups. The main result states how, considering the Zappa–Szép product of germs, we can "decompose" solutions into smaller ones :

Theorem ([Fei24a]). *Any finite solution can be constructed from solutions whose class is a prime power.*

These two statements aim to reduce the goal of classifying all finite braces to a simpler one:

First, the existence of the germs ([Deh15]) (who determine the solution), which are very particular braces, indicates that we can restrict to braces with "additive" group isomorphic to some $(\mathbb{Z}/d\mathbb{Z})^n$. The integer n determines the size of the solution and d its class, so it is fundamental to understand the behaviour of the class. In particular, having a sharp bound on d gives a large restriction to which braces should be considered for the classification, and this is precisely the point of the conjecture.

Then, the theorem indicates that one notion of "basic" solutions are the ones where the class d is a prime power. Another well-studied notion of "basic" solutions are indecomposable ones (where no proper subset of X is stable by r). Combining these two notions, it is shown in [Fei24a] that even more "fundamental" solutions should be the one where the size and the class are powers of the same prime. Moreover, it is also conjectured in [Fei24a] that in the case of indecomposable solutions, the class d is bounded by n. Putting all of this together, and assuming the two conjectures, one could hope to restrict to classifying braces with additive group $(\mathbb{Z}/p^a\mathbb{Z})^{p^b}$ where $a \leq b$.

3. Hecke algebras

Continuing on drawing parallels between structure groups and spherical Artin groups, one of my latest work consists in defining and studying a definition of Hecke algebras for solutions. In the case of spherical Artin groups, this is a sort of intermediate algebra between the ring algebras of the Artin group and the Coxeter group, from which many combinatorial and representation theoretical properties can be obtained (see for instance the book of Geck–Pfeiffer [GP00]). A very general definition is currently obtained and derived from a graphical interpretation of the structure group.

Let (X, r) be a solution of size n ($X = \{x_1, \ldots, x_n\}$), class d, structure group G and l-th germs $\overline{G}_l = G/(x_i^{[ld]})$, then we have the following:

Theorem ([Fei24b])**.** *Let $R = \mathbb{Z}[q_1^{\pm 1}, \ldots, q_l^{\pm 1}]$ and $P(X) = (X - q_1) \ldots (X - q_l) \in R[X]$. Then $\mathcal{H}(X, P) := R[G]/(P(x_i^{[d]})_{1 \leq i \leq n}$ has dimension equal to $\#\overline{G}_l = (dl)^n$.*

Moreover this algebra is absolutely semi-simple and $\iota\colon R \to \mathcal{R}$ sending q_i to q_i^{-1} extends to an anti-involution of $\mathcal{H}(X, P)$ by sending x_i to x_i^{-1}.

In particular, we can obtain an analogue of the generic Iwahori–Hecke algebra of a Coxeter group by taking $\mathcal{H} = \mathbb{Z}[q^{\pm 1}][G]/\langle x^{[2d]} = (q-1)x^{[d]} + q\rangle$.

In [To appear] we study properties of this algebra both via the usual framework of Iwahori–Hecke algebra, and via some classical constructions on solutions (relating a Hecke algebra of a solution to a Hecke algebra of its retraction).

Finally, Tits' Deformation theorem tells us that, over a suitable extension of the coefficients, this algebra is abstractly isomorphic to the group ring of $\overline{G}_l$. Thus, studying the character theory of this Hecke algebra could lead to understanding the character theory of the l-th germs (the "Coxeter-like groups"), in turn leading to a better comprehension of solutions.

4. Indecomposability and Irreducibility

A solution (X, r) to the Yang–Baxter equation is called decomposable if there exists a proper decomposition $X = Y \sqcup Z$ such that both $Y \times Y$ and $Z \times Z$ are stable under r, otherwise it is called indecomposable. In [Deh15] a monomial representation Θ (resp. $\overline{\Theta}_l$) of the structure group (resp. l-th germ) is defined, and was studied in [Fei24a]. One of the goals is to relate the indecomposability of a solution with properties of the monomial representation. In this sense, the following was obtained with Carsten Dietzel and Sylvia Properzi from VUB (Brussels) :

Theorem ([DFP24])**.** *Let (X, r) be a solution of class d. Let $l > 1$, then the following are equivalent :*

(1) (X, r) is indecomposable,
(2) Θ is irreducible,
(3) $\overline{\Theta}_l$ is irreducible.

For the case $l = 1$, the equivalence holds whenever either $d > 2$, or $d = 2$ and $\#\mathcal{G} < d^{\frac{n}{2}}$.

Moreover, the representation Θ (resp. $\overline{\Theta}_l$ for $l \geq 1$) are induced by a precise character of the stabilizer of any element of X.

When $l = 1$, $d = 2$ and $\#\mathcal{G} \geq d^{\frac{n}{2}}$, counterexamples are known to the theorem (the solution is indecomposable but $\overline{\Theta}_1$ is reducible). Understanding where those counterexamples come from, and classifying them, would lead to a better understanding of the monomial representation, indecomposable solutions and permutation groups.

5. Future work

In general, my interests lie in anything related to algebra: group theory, representation theory, combinatorics, homology theory, category theory, etc. and I try to make use of the different suitable techniques from these areas when needed. Thus, I would be greatly interested in working on new questions and collaborating with new people, as a way to continually learn and study mathematics while bringing new technics from one domain to another. In particular, here is a list of questions which I am currently working on:

5.1. **Bounding the class**

The goal would be to prove the conjecture on the sharp bounds of the class. For this, I have two different approaches:

a) Focusing on a brace approach of the permutation group, especially it's additive structure. This would involve giving an explicit characterization of its exponent, or the order of elements.
b) Showing the conjecture for indecomposable solutions ($d \leq n$) and understanding how decomposing a solution affects the class. This approach, may also lead to a better comprehension on how to construct decomposable solutions.

This work, and in general the study of solutions, is a great opportunity to work with other people. In particular, I've a lot of contact with Leandro Vendramin's team at VUB (Brussels, Belgium), and hope that this would be fruitful in producing new work.

5.2. **Studying the Hecke algebra**

It seems natural, after defining Hecke algebras, to define and study the Schur multiplier and the Kazhdan–Lusztig polynomials associated. This would be in the continuity of adapting the techniques from [GP00]. First restricting to particular families of solutions would be a first step. The difficult part in the general case would be that adapting the technics from [GP00] may be highly computational and more complicated (as we have more data than in the Coxeter case), but the proofs found so far are quite simpler (for instance of the Theorem in section 3), so the introduction of new tools and technics (in particular coming from the study of solutions) should make this project feasible.

Moreover, our construction is purely algebraic, but Hecke algebra are often approached geometrically. Finding a geometric interpretation of our construction (through a sort of "Bruhat-like" decomposition) would strengthen the parallel between Artin–Tits groups and Yang–Baxter structure groups.

Finally, it would be a very interesting application of Tits deformation theorem, to obtain the description of the field over which an Hecke algebra of a solution is separable (or even isomorphic to the group ring of the germ). Then to the study of the characters of this algebra, to understand the characters of the germ.

This would hopefully lead to collaboration with people working on Coxeter groups and/or Hecke algebras, as there is still a lot to be done.

5.3. **Generalization to Weyl groups**

The idea would be to adapt the techniques from [Deh15; Fei24a] to Weyl group (as the Yang–Baxter case is for the symmetric group). This would involve, the study of Algebraic Group and Geometric Group theory, in particular Weyl groups acting on maximal torus. In particular, this could also lead to a generalization of braces (replacing the additive abelian structure with a Coxeter one, a particular case of regular subgroup of the holomorphy group), and would be very interesting from a Garside perspective, as it would provide a Garside framework to encompass both Weyl groups and Yang–Baxter structure groups.

This question would be a nice opportunity to work with people focused on Geometric Group Theory as well as those working on braces. Moreover, this would directly lead to generalizations of all the questions mentioned above, thus a very large project encompassing many different objects and technics.

References

[AMV22] Ö Akgün, M. Mereb, and L. Vendramin. "Enumeration of Set-Theoretic Solutions to the Yang–Baxter Equation". In: *Math. Comp.* 91.335 (2022), pp. 1469–1481. DOI: 10.1090/mcom/3696.

[Bac18] David Bachiller. "Extensions, Matched Products, and Simple Braces". In: *Journal of Pure and Applied Algebra* 222 (2018), pp. 1670–1691. DOI: 10.1016/j.jpaa.2017.07.017.

[Cho10] Fabienne Chouraqui. "Garside Groups and Yang–Baxter Equation". In: *Communications in Algebra* 38.12 (2010), pp. 4441–4460. DOI: 10.1080/00927870903386502.

[Deh15] Patrick Dehornoy. "Set-Theoretic Solutions of the Yang–Baxter Equation, RC-calculus, and Garside Germs". In: *Advances in Mathematics* 282 (2015), pp. 93–127. DOI: 10.1016/j.aim.2015.05.008.

[DFP24] Carsten Dietzel, Edouard Feingesicht, and Silvia Properzi. *Indecomposability and Irreducibility of Monomial Representations for Set-Theoretical Solutions to the Yang-Baxter Equation*. Sept. 2024. DOI: 10.48550/arXiv.2409.10648. arXiv: 2409.10648.

[Dri92] V. G. Drinfeld. "On Some Unsolved Problems in Quantum Group Theory". In: vol. 1510. Lecture Notes in Mathematics. 1992, pp. 1–8. DOI: 10.1007/BFb0101175.

[ESS99] Pavel Etingof, Travis Schedler, and Alexandre Soloviev. "Set-Theoretical Solutions to the Quantum Yang-Baxter Equation". In: *Duke Mathematical Journal* 100 (1999), pp. 169–209. DOI: 10.1215/S0012-7094-99-10007-X.

[Fei24a] Edouard Feingesicht. "Dehornoy's Class and Sylows for Set-Theoretical Solutions of the Yang–Baxter Equation". In: *Int. J. Algebra Comput.* 34 (2024), pp. 147–173. DOI: 10.1142/S0218196724500048.

[Fei24b] Edouard Feingesicht. *Hecke Algebras for Set-Theoretical Solutions to the Yang–Baxter Equation*. Nov. 2024. DOI: 10.48550/arXiv.2411.00695. arXiv: 2411.00695.

[Gat04] Tatiana Gateva-Ivanova. "A Combinatorial Approach to the Set-Theoretic Solutions of the Yang–Baxter Equation". In: *Journal of Mathematical Physics* 45 (2004), pp. 3828–3858. DOI: 10.1063/1.1788848.

[GP00] Meinolf Geck and Götz Pfeiffer. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*. Clarendon Press, 2000. ISBN: 978-0-19-850250-0.

[GV98] Tatiana Gateva-Ivanova and Michel Van den Bergh. "Semigroups of I-Type". In: *Journal of Algebra* 206 (1998), pp. 97–112. DOI: 10.1006/jabr.1997.7399.

[Rum05] Wolfgang Rump. "A Decomposition Theorem for Square-Free Unitary Solutions of the Quantum Yang-Baxter Equation". In: *Advances in Mathematics* 193 (2005), pp. 40–55. DOI: 10.1016/j.aim.2004.03.019.

[Rum07] Wolfgang Rump. "Braces, Radical Rings, and the Quantum Yang–Baxter Equation". In: *Journal of Algebra* 307.1 (Jan. 2007), pp. 153–170. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2006.03.040.